Problems to Effective theories for heavy quarks

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Problem 1: Generalities

a) Knowing that the action in D space-time dimensions $\int \mathcal{L}(x)\mathrm{d}^D x$, should be dimensionless, calculate the dimensions of the quark, gluon and ghost fields, as well as of the gauge coupling g, in units of mass or momentum.

b) With the result of a), calculate the mass dimension of the free momentum space quark and gluon propagators $S^{(0)}(p)$ and $D^{(0)}_{\mu\nu}(k)$ respectively.

c) Consider the Dirac-Algebra in an arbitrary dimension D

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbb{I}$$

with the space-time indices μ, ν running from $0, 1, 2, \ldots, D-1$. Furthermore:

$$g_{\mu\nu}g^{\mu\nu} = D \quad \text{and} \quad \gamma_\mu\gamma^\mu = D\mathbb{I}.$$

Explicitly perform the contractions in the Dirac-matrix expressions

- $\gamma_\mu\gamma_\nu\lambda\gamma^\mu$,
- $\gamma_\mu\gamma_\nu\gamma^\mu$.

Problem 2: Feynman Integrals in D dimensions

The goal of this problem is the continuation of the Feynman Integrals, which arise in the calculations to an arbitrary dimension.

Consider the following integral in D dimensions:

$$\tilde{I}(1,1;q^2,m^2) \equiv \mu^{2\epsilon} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{[p^2 - m^2 + i0][(q-p)^2 - m^2 + i0]}.$$

A convenient way of proceeding to calculate such type of integrals is due to Feynman and consists in rewriting the integrand with the help of the following formula:

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax + b(1-x)]^{\alpha+\beta}} \tag{1}$$

where $\Gamma(z)$ denotes the Eulers Γ-function.

a) Explicitly verify relation (1) for the case $\alpha = 1$ and $\beta = 2$.

b) Show that with the Feynman parametrisation of eq. (1), the integral $\tilde{I}(1,1;q^2,m^2)$ can be brought into the from

$$\tilde{I}(1,1;q^2,m^2) = \mu^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{[k^2 - a^2]^2},$$

with $a^2 \equiv m^2 - x(1-x)q^2$.

c) We shall consider a generalization of this type of integral. Let

$$I(\alpha,\beta;a^2) \equiv \mu^{2\epsilon} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^{2\alpha}}{[k^2 - a^2 + i0]^\beta}. \tag{2}$$

Derive

$$I(\alpha,\beta;a^2) = \frac{i}{(4\pi)^2}[-a^2]^{\alpha-\beta+2} \left(\frac{4\pi\mu^2}{a^2}\right)^\epsilon \frac{\Gamma(\beta-\alpha-2+\epsilon)\Gamma(\alpha+2-\epsilon)}{\Gamma(\beta)\Gamma(2-\epsilon)}$$

from (2).

Note that the dimension D only appears in exponents and the γ-functions, which allows to analytically continue the result also to non-integer D. The result for $I(0,2;a^2)$ can now be employed in order to continue with the calculation of $\tilde{I}(1,1;q^2,m^2)$. The final result one obtains is

$$\tilde{I}(1,1;q^2,m^2) = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\hat{\epsilon}} - \ln\frac{m^2}{\mu^2} + 2 - \sqrt{1 - \frac{4m^2}{q^2}} \ln \frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{\sqrt{1-\frac{4m^2}{q^2}}-1} \right\},$$

which is known as the scalar one-loop integral with equal masses.

Problem 3: Quark selfenergy

Derive the quark selfenergy $\Sigma(p)$ at order g^2.

a) Therefore show that

$$\Sigma^{(1)}(p) = ig^2 C_F \int \frac{\mathrm{d}^D k}{(2\pi)^D} [\gamma^\mu S^{(0)}(p-k)\gamma^\nu] D^{(0)}_{\mu\nu}(k), \tag{3}$$

follows from the perturbative correction to the quark propagator in momentum space:

$$S_{ij}(p) = -i \int \mathrm{d}^D x \mathrm{e}^{ipx} \langle 0 | T\{q_i(x)\bar{q}_j(0)\mathrm{e}^{i\int \mathrm{d}^D z \mathcal{L}_I(z)}\} | 0 \rangle,$$

where i, j denote colour indices and we have suppressed the spinor indices and $\mathcal{L}_I(z)$ is the interaction part of the Lagrangian.

b) To proceed with the calculation of the selfenergy $\Sigma(p)$ we insert the free quark and gluon propagators into (3):

$$\Sigma^{(1)}(p) = -ig^2\mu^{-2\epsilon} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\gamma^\mu(\not{p} - \not{k} + m)\gamma^\nu}{k^2[(p-k)^2 - m^2]} \left[g_{\mu\nu} - (1-a)\frac{k_\mu k_\nu}{k^2} \right]$$

Calculate $\Sigma^{(1)}(p)$ in the Feynman gauge ($a = 1$) and to linear order in the quark mass m. This is sufficient to determine the renormalisation constant Z_m to one-loop.

Problem 4:

a) Recalling that the renormalisation constant for the quark mass Z_m is given by

$$Z_m = 1 - \frac{3}{4} C_F a_s \frac{1}{\hat{\epsilon}} + \mathcal{O}(a_s^2),$$

calculate the one-loop coefficient γ_1 for the mass anomalous dimension

$$\gamma(a_s) = -\frac{\mu}{m}\frac{\mathrm{d}m}{\mathrm{d}\mu} = \gamma_1 a_s + \gamma_2 a_s^2 \gamma_3 a_s^3 + \gamma_4 a_s^4 + \ldots$$

b) Explicitly calculate the renormalisation group running for the quark mass to one-loop order.

Solutions to Effective theories for heavy quarks

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Problem 1: Generalities

Throughout, we employ natural units with $c \equiv 1$ and $\hbar \equiv 1$.

a) If the operator dim[...] represents the mass dimension of the quantity under investigation, one finds, using $D = 4 - 2\epsilon$:

$$\begin{aligned}
\dim[q_i^A(x)] &= \frac{1}{2}(D-1) = \frac{3}{2} - \epsilon \\
\dim[B_\mu^a(x)] &= \frac{D}{2} - 1 = 1 - \epsilon \\
\dim[c^a(x)] &= \frac{D}{2} - 1 = 1 - \epsilon \\
\dim[g] &= 2 - \frac{D}{2} = \epsilon
\end{aligned}$$

b)

$$\begin{aligned}
\dim[S^{(0)}(p)] &= -1 \\
\dim[D_{\mu\nu}^{(0)}(k)] &= -2
\end{aligned}$$

Thus the momentum space propagators retain the same mass dimension as in 4 space-time dimensions.

c)

$$\gamma_\mu\gamma_\nu\gamma_\lambda\gamma^\mu = 2\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda\gamma^\mu = 4g_{\mu\nu}\mathbb{I} + (D-4)\gamma_\nu\gamma_\lambda$$

$$\gamma_\mu\gamma_\nu\gamma^\mu = (2-D)\gamma_\nu$$

Problem 2: Feynman Integrals in D dimensions

a) The solution is straightforward:

$$\begin{aligned}
\frac{1}{ab^2} &= 2\int_0^1 \frac{(1-x)\mathrm{d}x}{[ax+b(1-x)]^3} = 2\int_0^1 \frac{z\mathrm{d}z}{[a+(b-a)z]^3} \\
&= \frac{2}{(b-a)}\int_0^1 \frac{[a+(b-a)z-a]}{[a+(b-a)z]^3}\mathrm{d}z \\
&= \frac{2}{(b-a)}\left[\int_0^1 \frac{\mathrm{d}z}{[a+(b-a)z]^2} - \int_0^1 \frac{a\mathrm{d}z}{[a+(b-a)z]^3}\right] \\
&= \frac{2}{(b-a)^2}\left[\left.\frac{-1}{[a+(b-a)z]}\right|_0^1 + \left.\frac{a}{2[a+(b-a)z]^2}\right|_0^1\right] \\
&= \frac{2}{(b-a)^2}\left[\frac{1}{a}-\frac{1}{b}+\frac{a}{2b^2}-\frac{1}{2a}\right] = \frac{1}{ab^2}
\end{aligned}$$

Making use of the hypergeometric function ${}_2F_1(a,b;c;z)$, the relation can also be shown in the general case:

$$\begin{aligned}
\frac{1}{a^\alpha b^\beta} &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \mathrm{d}x\frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{b^{\alpha+\beta}}\int_0^1 \mathrm{d}x x^{\alpha-1}(1-x)^{\beta-1}\left[1-\left(1-\frac{a}{b}\right)x\right]^{-\alpha-\beta} \\
&= \frac{1}{b^{\alpha+\beta}}{}_2F_1\left(\alpha+\beta,\alpha;\alpha+\beta;1-\frac{a}{b}\right) = \frac{1}{b^{\alpha+\beta}}\left(\frac{b}{a}\right)^\alpha = \frac{1}{a^\alpha b^\beta}
\end{aligned}$$

b)

$$\begin{aligned}
\tilde{I}(1,1;q^2,m^2) &\equiv \mu^{2\epsilon}\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{1}{[p^2-m^2+i0][(q-p)^2-m^2+i0]} \\
&= \mu^{2\epsilon}\int_0^1 \mathrm{d}^Dx\int\frac{\mathrm{d}^Dp}{(2\pi)^D}\frac{1}{[ax+b(1-x)]^2}
\end{aligned}$$

with $\alpha=\beta=1$, $a=p^2-m^2+i0$ and $b=(q-p)^2-m^2+i0$. When the square is completed, we change the momentum integration variable according to

$$p \to k+q(1-x).$$

Thus we get

$$\begin{aligned}
\tilde{I}(1,1;q^2,m^2) &= \mu^{2\epsilon}\int_0^1 \mathrm{d}^Dx\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2-m^2+q^2x-q^2x^2} \\
&= \mu^{2\epsilon}\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{[k^2-a^2]^2}
\end{aligned}$$

with $a^2 \equiv m^2 - x(1-x)q^2$.

c) The first step of solving the integral consists in performing a Wick rotation, which amounts to rotating the time-axis k^0 into an imaginary time direction ik^D. Then we find:

$$
\begin{aligned}
I(\alpha,\beta;a^2) &= \mu^{2\epsilon}\int \frac{\mathrm{d}k^0\mathrm{d}k^1\ldots\mathrm{d}k^{D-1}}{(2\pi)^D}\frac{\left[(k^0)^2-\sum_{i=1}^{D}(k^i)^2\right]^\alpha}{\left[(k^0)^2-\sum_{i=1}^{D}(k^i)^2-a^2+i0\right]^\beta} \\
&= i(-1)^{\alpha-\beta}\mu^{2\epsilon}\int \frac{\mathrm{d}k^1\ldots\mathrm{d}k^{D-1}\mathrm{d}k^D}{(2\pi)^D}\frac{\left[(k^D)^2+\sum_{i=1}^{D}(k^i)^2\right]^\alpha}{\left[(k^D)^2+\sum_{i=1}^{D}(k^i)^2+a^2\right]^\beta} \\
&= i(-1)^{\alpha-\beta}\mu^{2\epsilon}\int \frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{k^{2\alpha}}{[k^2+a^2]^\beta},
\end{aligned}
$$

where the last integral has to be performed in an D-dimensional Euclidean space. Since it only depends on k^2, this integration is best performed in D-dimensional polar coordinates. This yields

$$I(\alpha,\beta;a^2) = i(-1)^{\alpha-\beta}\frac{\mu^{2\epsilon}}{(2\pi)^D}\int_0^\infty \mathrm{d}k\int \mathrm{d}\Omega\frac{k^{2\alpha+D-1}}{[k^2+a^2]^\beta}.$$

The angular integration results in the area of a D-dimensional unit-sphere which is given by

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}.$$

Finally, we are left with the remaining radial integration. The idea behind solving this integral is to rewrite it in terms of an integral which is Eulers β-function $B(u,v)$ defined by:

$$B(u,v) \equiv \int_0^1 z^{u-1}(1-z)^{v-1}\mathrm{d}z = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

This transformation can be achieved with the substitution

$$k^2 = \frac{a^2(1-z)}{z}$$

where z runs from 0 to 1. One then finds:

$$
\begin{aligned}
I(\alpha,\beta;a^2) &= i(-1)^{\alpha-\beta}\mu^{2\epsilon}\frac{[a^2]^{\frac{D}{2}+\alpha-\beta}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\beta-\alpha-\frac{D}{2})\Gamma(\alpha+\frac{D}{2})}{\Gamma(\beta)\Gamma(\frac{D}{2})} \\
&= \frac{i}{(4\pi)^2}[-a^2]^{\alpha-\beta+2}\left(\frac{4\pi\mu^2}{a^2}\right)^\epsilon\frac{\Gamma(\beta-\alpha-2+\epsilon)\Gamma(\alpha+2-\epsilon)}{\Gamma(\beta)\Gamma(2-\epsilon)}.
\end{aligned}
$$

Problem 3: Quark selfenergy

a) We start with

$$S_{ij}(p) = -i\int \mathrm{d}^D x \mathrm{e}^{ipx}\langle 0|T\{q_i(x)\bar{q}_j(0)\mathrm{e}^{i\int \mathrm{d}^D z \mathcal{L}_I(z)}\}|0\rangle,$$

where we have suppressed the the spinor indices. The only piece of the interaction Lagrangian $\mathcal{L}_I(z)$ which contributes at this order is the quark-gluon interaction. We need this term twice, since the vacuum-expectation value of a single gluon field vanishes. Thus we find

$$S^{(1)}_{ij}(p) = ig^2\int \mathrm{d}^D x\int \mathrm{d}^D z_1\int \mathrm{d}^D z_2 \mathrm{e}^{ipx}\left[S^{(0)}(x-z_1)\gamma^\mu\frac{\lambda^a}{2}S^{(0)}(z_1-z_2)\gamma^\nu\frac{\lambda^a}{2}S^{(0)}(z_2)\right]_{ij} \times D^{(0)}_{\mu\nu}(z_1-z_2).$$

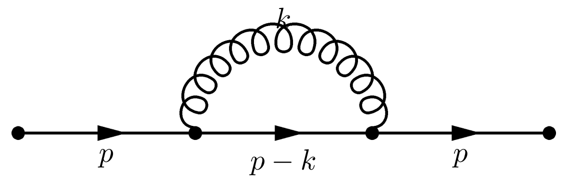
The next step consists in replacing the coordinate-space propagators through their momentum-space representation. Then the coordinate-space integrations can be carried out, yielding δ-distributions, which allow to perform three of the momentum-space integrals. This leads to

$$S^{(1)}_{ij}(p) = ig^2 C_F\delta_{ij}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\left[S^{(0)}(p)\gamma^\mu S^{(0)}(p-k)\gamma^\nu S^{(0)}(p)\right]D^{(0)}_{\mu\nu}(k),$$

where we have used the fact that

$$\left[\frac{\lambda^a}{2}\frac{\lambda^a}{2}\right]_{ij} = \frac{N_C^2-1}{2N_C}\delta_{ij} = C_F\delta_{ij} \overset{N_C=3}{=} \frac{4}{3}\delta_{ij},$$

with C_F being the Casimir operator of the gauge group $SU(N_C)$ in the fundamental representation. In the diagrammatic language of Feynman diagrams, this would correspond to:



Now, we rewrite the quark propagator in the following form:

$$S_{ij}(p) = \delta_{ij}S^{(0)}(p) + \delta_{ij}S^{(0)}(p)\Sigma(p)S^{(0)}(p) + \ldots$$

The new function $\Sigma(p)$ is called the quark selfenergy, and at order g^2 takes the form

$$\Sigma^{(1)}(p) = ig^2 C_F\int\frac{\mathrm{d}^D k}{(2\pi)^D}[\gamma^\mu S^{(0)}(p-k)\gamma^\nu]D^{(0)}_{\mu\nu}(k).$$

b) For the calculation of the quark selfenergy, we need two types of massless Feynman integrals, which are presented in the general case first:

$$\hat{I}(\alpha,\beta;p) \equiv \mu^{2\epsilon}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[k^2+i0]^\alpha[(p-k)^2+i0]^\beta}$$
$$= \frac{i}{4\pi}\left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon \frac{1}{p^{2(\alpha+\beta-2)}}\frac{\Gamma(2-\alpha-\epsilon)\Gamma(2-\beta-\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(4-\alpha-\beta-2\epsilon)}\Gamma(\alpha+\beta-2+\epsilon)$$

$$\hat{I}_\mu(\alpha,\beta;p) \equiv \mu^{2\epsilon}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{k_\mu}{[k^2+i0]^\alpha[(p-k)^2+i0]^\beta}$$
$$= p_\mu\frac{(2-\alpha-\epsilon)}{(4-\alpha-\beta-2\epsilon)}\hat{I}(\alpha,\beta;p)$$

The first expression can be derived by introducing Feynman parameters and making use of the result of Problem 2 c) for $I(\alpha,\beta,a^2)$. The remaining x-integration can be performed analogously to the derivation of this formula.

To derive the second expression, one can use a trick which is quite useful in general. Since the result can only depend on the momentum p, and must transform like a Lorentz vector, it must have the structure $A(p^2)p_\mu$. Now, we can multiply both sides with $2p^\mu$, yielding

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{2p\cdot k}{[k^2+i0]^\alpha[(p-k)^2+i0]^\beta} = 2p^2A(p^2).$$

The numerator under the integral can be rewritten as

$$2p\cdot k = p^2+k^2-(p-k)^2,$$

which results in three integrals of the type $\hat{I}(\alpha,\beta;p)$. After some massaging of the Γ-functions, one arrives at the given integral formula for $\hat{I}_\mu(\alpha,\beta;p)$.

The two special cases which are needed for the calculation of $\Sigma^{(1)}(p)$ are:

$$\hat{I}(1,1;p) = \frac{i}{4\pi^2}\left(\frac{4\pi\mu^2}{-p^2}\right)\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\Gamma(\epsilon)$$

and

$$\hat{I}_\mu(1,1;p) = \frac{p_\mu}{2}\hat{I}(1,1;p).$$

This can be inserted into our expression for $\Sigma^{(1)}(p)$:

$$\Sigma^{(1)}(p) = -ig^2\mu^{-2\epsilon}C_F\mu^{2\epsilon}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma^\mu}{k^2[(p-k)^2-m^2]}$$
$$= ig^2\mu^{-2\epsilon}C_F\left[(1-\epsilon)\not{p}-(4-2\epsilon)m\right]\hat{I}(1,1;p)$$
$$= \frac{g^2\mu^{-2\epsilon}}{(4\pi)^2}C_F\left[(-1+\epsilon)\not{p}+(4-2\epsilon)m\right]\left\{\frac{1}{\hat{\epsilon}}-\ln\frac{-p^2}{\mu^2}+2+\mathcal{O}(\epsilon)\right\}$$

Splitting up $\Sigma(p)$ in the form

$$\Sigma(p) = \not{p}\Sigma_p(p) + m\Sigma_m(p),$$

one finds:

$$\Sigma_p^{(1)}(p) = \frac{C_F}{4}\frac{\alpha_s}{\pi}\left\{-\frac{1}{\hat{\epsilon}} + \ln\frac{-p^2}{\mu^2} - 1\right\}$$

and

$$\Sigma_m^{(1)}(p) = \frac{C_F}{4}\frac{\alpha_s}{\pi}\left\{\frac{4}{\hat{\epsilon}} - 4\ln\frac{-p^2}{\mu^2} + 6\right\},$$

where we have defined the dimensionless coupling

$$\alpha_s \equiv \frac{g^2\mu^{-2\epsilon}}{(4\pi)}.$$

Problem 4:

a) We start with the relation between the bare and renormalized quark mass,

$$m = Z_m^{-1}m^0$$

where m^0 corresponds to the bare quark mass. Inserting this in the definition of the mass anomalous dimension yields:

$$\begin{aligned}\gamma(a_s) &= -\frac{\mu}{m}\frac{\mathrm{d}m}{\mathrm{d}\mu} = -\frac{\mu}{Z_m^{-1}m^0}\frac{\mathrm{d}(Z_m^{-1}m^0)}{\mathrm{d}\mu} = \frac{\mu}{Z_m}\frac{Z_m}{\mathrm{d}\mu} = \frac{\mu}{Z_m}\frac{\mathrm{d}a_s}{\mathrm{d}\mu}\frac{Z_m}{\mathrm{d}a_s} \\ &= -\frac{\beta(a_s)}{Z_m}\frac{Z_m}{\mathrm{d}a_s} = -2a_s\epsilon\left(-\frac{3}{4}\right)C_F\frac{1}{\hat{\epsilon}} + \mathcal{O}(a_s^2) = \frac{3}{2}C_F a_s + \mathcal{O}(a_s^2)\end{aligned}$$

where we have used the renormalisation constant $Z_m^{(1)} = \left(-\frac{3}{4}\right)C_F\frac{1}{\hat{\epsilon}}$. Thus we obtain

$$\gamma_1 = \frac{3}{2}C_F = 2,$$

with $N_C = 3$.

b) From integrating the renormalization group equation for the quark mass by separation of variables follows

$$\int_{m(\mu_1)}^{m(\mu_2)}\frac{\mathrm{d}m}{m} = \ln\frac{m(\mu_2)}{m(\mu_1)} = -\int_{\mu_1}^{\mu_2}\frac{\mathrm{d}\mu}{\mu}\gamma(a_s) = \int_{a_s(\mu_1)}^{a_s(\mu_2)}\mathrm{d}a_s\frac{\gamma(a_s)}{\beta(a_s)}.$$

Thus we obtain

$$m(\mu_2) = m(\mu_1) \exp\left\{ \int_{a_s(\mu_1)}^{a_s(\mu_2)} \mathrm{d}a_s \frac{\gamma(a_s)}{\beta(a_s)} \right\}.$$

At one-loop order, the exponential factor for the running of the quark mass takes the form

$$\begin{aligned} \exp\left\{ \int_{a_s(\mu_1)}^{a_s(\mu_2)} \mathrm{d}a_s \frac{\gamma(a_s)}{\beta(a_s)} \right\} &= \exp\left\{ \frac{\gamma_1}{\beta_1} \int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{\mathrm{d}a_s}{a_s} \right\} \\ &= \exp\left\{ \frac{\gamma_1}{\beta_1} \ln \frac{a_s(\mu_2)}{a_s(\mu_1)} \right\} = \left(\frac{a_s(\mu_2)}{a_s(\mu_1)} \right)^{\frac{\gamma_1}{\beta_1}}. \end{aligned}$$

The running of the quark mass is then given by

$$m(\mu_2) = m(\mu_1) \left(\frac{a_s(\mu_2)}{a_s(\mu_1)} \right)^{\frac{\gamma_1}{\beta_1}} \left[1 + \mathcal{O}(a_s)\right].$$